DISSERTATIO MATHEMATICA,

DE

LINEIS CURVIS PARALLELIS.

Cujus

Partem Priorem

Conſ. Ampliſſ. Facult. Philoſ. Aboënſ.

PRÆSIDE

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ABOÆ, typis Frenckellianis.

5.

§. I.

Quod in Theoria linearum rectarum parallelarum assumi solet principium, lineas videlicet rectas, manente distantia inter easdem invariata, esse parallelas, in genere lineis de curvis non valet. Harum quidem parallelismus tangentium ope ita determinatur, ut, si ductæ ad puncta quævis correspondentia lineæ tangentes inter se sint parallelæ, ipsas quoque curvas in iisdem punctis parallelas esse dicamus. Duplici autem modo puncta ista correspondentia determinari possunt; locus enim ipsorum vel in communi curvarum linea Normali esse potest, vel etiam in linea ordinatim applicata. In casu priori distantia inter curvas est invariata; in altero vero inæqualiter a se invicem distent curvæ, necesse est, quo in casu curvæ similes appellantur. Hinc itaque sequitur, ut parallelismus curvarum duplici modo concipi possit: aut enim distantia inter curvas in quibusvis ipsarum punctis eadem est, aut hæc distantia invariata non manet; parallelæ tamen dicuntur curvæ, eo ex fundamento, quod tangentes curvarum in utroque casu æqualiter a se invicem distare possunt.

Sed quæ jam de curvis in genere sunt allata, de Circulis pariter dici non possunt, quippe qui ejus sunt indolis, ut omues sint similes & insimul, si paralleli fuerint, æqualibus in quovis puncto intervallis distent.

Theoriam curvarum parallelarum, quamvis insignis ipsius sit usus, nullus, quantum nobis quidem constat, ante tempora Cel. J. G. KÆSTNER tradidit. Maxime autem inclytus hic Geometra, in litteris ad Cl. R WOLTMAN datis, quæ in *Beyträge zur hydraulischen architectur, aufgesezt von* R. WOLTMAN, *2:er Band*, *Götting.* 1792, pag. 33-57 recensentur, hanc ingeniose enodavit rem, ostendens, quatuor adhiberi posse methodos lineam curvam datæ cuidam parallelam ita ducendi, ut distantia inter illas semper eadem maneat, quarum methodorum illam in sequentibus paullo fusius exponere statuimus, qua relatio inter Coordinatas Orthogonales quæritur ex data æquatione illius curvæ, cui parallela est ducenda, L. B. censuræ jam submittentes, quæ ad hanc illustrandam apta duximus.

§. 2.

Quamvis primo intuitu videatur, æquationes pro lineis curvis datæ cuidam parallelis facillime inveniendas esse, si quantitates constantes, relationem inter coordinatas curvæ datæ Orthogonales determinantes, alia quantitate cognita augeantur vel minuantur; minime tamen hac methodo pro quibusvis curvis uti possumus. Æquatio enim Circuli hoc modo determinari potest, ita ut datæ hujusmodi curvæ parallela inveniatur; de reliquis vero generatim non valet. Sic, ex. gr. si in Ellipsi, cujus æquatio est $y^2 = b^2 - \frac{b^2 x^2}{a^2}$, sumta origine abscissarum in centro, loco axium a & b statuantur $a \pm c$ & $b \pm c$, nova quidem exsurgit æquatio, naturam Ellipsis ejusmodi exhibens, quæ a data, in ipsis verticibus axium, æquali-

bus

bus distat intervallis, non autem in quibusvis aliis punctis ubicunque sumtis, quod haud difficile est demonstratu.

Sit enim *C* centrum Ellipsium *AM* atque *BQ*, quarum axes sunt a & b, $a - c = \alpha$ & $b - c = \beta$ respective, $CP = x$ & $CR = z$; ducta, porro, linea Normalis *MN*, perspicuum est, si curvæ æquidistantes forent, esse $MQ = AB = c$; demittantur lineæ *MP* & *QR* perpendiculariter in *CA* & ducatur *QS* parallela ipsi *CP*, erit ob $\triangle MPN \backsim \triangle QRN$, $MN : QN :: PN : RN$, adeoque etiam $MN^2 : QN^2 :: PN^2 : RN^2$, seu $\frac{b^2}{a^4}$ $(a^4 + (b^2 - a^2)\,x^2) : \frac{\beta^2}{\alpha^4}\,(\alpha^4 + (\beta^2 - \alpha^2)\,z^2) :: \frac{b^4 x^2}{a^4} :$ $\frac{\beta^4\,z^2}{\alpha^4}$, sumtis valoribus normalium atque subnormalium in utraque Ellipsi; unde $(a^4 + (b^2 - a^2)\,x^2)\,\beta^2 z^2 =$ $(\alpha^4 + (\beta^2 - \alpha^2)\,z^2)\,b^2 x^2$ & $z^2 = \frac{\alpha^4 b^2 x^2}{(a^4 + (b^2 - a^2)x^2)\beta^2 - (\beta^2 - \alpha^2)b^2 x^2}$ adeoque $z = \frac{\alpha^2 bx}{((a^4 + (b^2 - a^2)x^2)\beta^2 - (\beta^2 - \alpha^2)b^2 x^2)^{\frac{1}{2}}} =$ $\frac{\alpha^2 bx}{(a^4\beta^2 - x^2(a^2\beta^2 - \alpha^2 b^2))^{\frac{1}{2}}}$ facta debita reductione. Est autem $\triangle PMN \backsim \triangle MSQ$, & hinc $PN : MN :: SQ : MQ$, seu $\frac{b^2 x}{a^2} : \frac{b}{a^2}\sqrt{a^4 + (b^2 - a^2)\,x^2} :: x - z = x -$

 $\alpha^2 bx$

$$\frac{\alpha^2\, b x}{(a^4\, \beta^2 - x^2\,(a^2\, \beta^2 - \alpha^2\, b^2))^{\frac{1}{2}}} : MQ =$$

$$\frac{(a^4 + (b^2 - a^2)x^2)^{\frac{1}{2}}\,(a^4\,\beta^2 - x^2(a^2\beta^2 - \alpha^2 b^2))^{\frac{1}{2}} - \alpha^2 b^2)}{b\,(a^4\beta^2 - x^2\,(a^2\beta^2 - \alpha^2 b^2))^{\frac{1}{2}}},$$

unde, restitutis valoribus $\alpha = a - c$ & $\beta = b - c$, videtur lineam istam MQ quantitati c æqualem non esse, adeoque nec curvas æqualibus distare intervallis.

Hoc vero præterea exinde patet, quod, posito $x = 0$, esset $MQ = \frac{a^2}{b} - \frac{\alpha^2}{\beta}$, quum tamen hoc in casu æqualis c esse debuisset; quibus intelligitur lineam MN, Normalem ipsi AM, perpendiculariter in arcum BQ non insistere, quare nec linea MQ minima inter curvas esse potest distantia, nec æqualis quantitati c constanti, qua axes Ellipsis BQ diminuti sunt. Unicus tamen adest casus, quo Ellipses AM & BQ eandem lineam normalem habere possunt, assumta videlicet $c = \frac{\overline{a-b}^2 + b}{b}$, quod ex æquatione $MQ = \frac{a^2}{b} - \frac{\overline{a-c}^2}{b-c}$ facillime deducitur.

§. 3.

Solutionis vero problematis nostri instituendæ, curvam scilicet, datæ cuidam parallelam ducere, sequens nobis commodissima videtur methodus. Sumto in Curva AM dato puncto quodam M, cujus linea Normalis in isto puncto sit MN, axem abscissarum AC in N secans, capiatur

tur $MQ = c = AB =$ distantiæ inter curvas invariatæ, & ducantur PM atque RQ perpendiculariter in AC. Sit $AP = x$, $PM = y$, $BR = z$ & $RQ = v$; erit, ob $\triangle PMN \backsim RQN$, $MN : PN :: QN : RN$, seu, adhibitis harum linearum valoribus generalibus, $\frac{y\sqrt{dx^2 + dy^2}}{dx}$: $\frac{ydy}{dx} :: \frac{y\sqrt{dx^2 + dy^2}}{dx} - c : RN = \frac{ydy}{dx} - \frac{cdy}{\sqrt{dx^2 + dy^2}}$, adeoque $BR = AP + PN - AB - RN = z = x + \frac{ydy}{dx} - \frac{ydy}{dx} + \frac{cdy}{\sqrt{dx^2 + dy^2}} - c = x - c + \frac{cdy}{\sqrt{dx^2 + dy^2}}$. Est autem $PN : PM :: dy : dx :: RN : RQ :: \frac{ydy}{dx} - \frac{cdy}{\sqrt{dx^2 + dy^2}} : v = y - \frac{cdx}{\sqrt{dx^2 + dy^2}}$, unde patet relationem inter coordinatas z & v Orthogonales inveniri posse ex data æquatione inter x & y.

COROLL. Existente Curva AM Algebraica, erit semper æquatio inter z & v Algebraica; quod ex ipsa inspectione æquationum $z = x - c + \frac{cdy}{\sqrt{dx^2 + dy^2}}$ & $v = y - \frac{cdx}{\sqrt{dx^2 + dy^2}}$ videri potest. Quod si vero Tranſcendens curva fuerit data, æquatio quoque inter v & z tranſcendens evadet neceſſe eſt, cujus exemplum WOLTMAN l. c. exhibet.

SCHOL. Eandem plane methodum, qua uſi ſumus in determinanda inter coordinatas orthogonatas curvæ BQ, datæ AM parallelæ, relatione in caſu, quo intra limites curvæ datæ cadit, facile etiam in quolibet alio caſu, obſervatis ſolummodo variationibus ſignorum, adhiberi poſſe, perſpicuum eſt. Sic, ſi concipiatur punctum B, ad alteram partem ipſius puncti A, reſpectu axis abſciſſarum AP, determinatum, ita, ut ſit $AB = c$; erit hoc in caſu $z = x + c - \frac{c\,dy}{\sqrt{dx^2 + dy^2}}$ & $v = y + \frac{c\,dx}{\sqrt{dx^2 + dy^2}}$.

EXEMPL. Quod ſi ſit propoſita æquatio curvæ datæ AM, $y^2 = A + Bx + Cx^2$, quæ naturam Sectionum Conicarum generatim exhibet; erit $y = \sqrt{A + Bx + Cx^2}$, atque $dy = \frac{dx(\frac{1}{2}B + Cx)}{\sqrt{A + Bx + Cx^2}}$, & $\sqrt{dx^2 + dy^2} = dx\frac{((\frac{1}{2}B + Cx)^2 + A + Bx + Cx^2)^{\frac{1}{2}}}{\sqrt{A + Bx + Cx^2}}$; ex quibus habebitur $z = x - c + \frac{c\,dy}{\sqrt{dx^2 + dy^2}} = x - c +$

$$\frac{c(\frac{1}{2}B+Cx)}{((\frac{1}{2}B+Cx)^2+A+Bx+Cx^2)^{\frac{1}{2}}} \quad \& \quad v = y - \frac{cdy}{\sqrt{dx^2+dy^2}}$$

$$= \sqrt{A+Bx+Cx^2} - \frac{c\sqrt{A+Bx+Cx^2}}{((\frac{1}{2}B+Cx)^2+A+Bx+Cx^2)^{\frac{1}{2}}}.$$ Quo jam relatio inter z & v innotescat, exterminanda est quantitas x; ad hunc autem finem obtinendum, tollenda est irrationalitas æquationum allatarum: at, quum absque prolixo admodum calculo fieri nequeat, seorsim pro sectionibus Conicis in sequentibus relationem istam quærere nobis proposuimus.

§. 4.

Existente arcu AM portione Circuli, cujus æquatio est $y^2 = ax - x^2$, sumta origine abscissarum in vertice Diametri a; erit in formula generali $A = 0$, $B = a$ & $C = -x$ adeoque $z = x - c + \frac{c(\frac{1}{2}B+Cx)}{((\frac{1}{2}B+Cx)^2+A+Bx+Cx^2)^{\frac{1}{2}}}$

$= x - c + \frac{c(\frac{1}{2}a - x)}{\frac{1}{2}a}$, unde $x = \frac{az}{a-2c}$. Erat autem

$$v = \sqrt{A+Bx+Cx^2} - \frac{c\sqrt{A+Bx+Cx^2}}{((\frac{1}{2}B+Cx)^2+A+Bx+Cx^2)^{\frac{1}{2}}}$$

$= \sqrt{ax-x^2} - \frac{c\sqrt{ax-x^2}}{\frac{1}{2}a}$, quibus itaque eruitur a^2v^2

$= (a-2c)^2\,(ax-x^2)$, seu $x.\overline{a-x} = \frac{a^2v^2}{(a-2c)^2}$, facta

debi-

debita terminorum reductione. Substituto vero in hac æquatione valore ipsius x antea jam determinato, habebitur

$$\frac{az}{(a-2c)}\left(\overline{a.a-2c}-\frac{az}{(a-2c)}\right)=\frac{a^2v^2}{(a-2c)^2}, \text{ seu } \frac{az}{(a-2c)^2}$$

$$(\overline{a.a-2c}-z)=\frac{a^2v^2}{(a-2c)^2},$$ & ducta æquatione in $(a-2c)^2$, eademque per a^2 divisa prodit $v^2=z\,a-2c-z)=az-2cz-z^2$, quæ quidem est æquatio ad circulum cujus Diameter $=a-2c$. Hinc itaque luculenter perspicitur illa Circuli proprietas, cujus mentionem supra §. 1. fecimus, Circulum videlicet ejus esse indolis ut Curva illi parallela semper sit similis & insimul æquidistans.

§: 5.

In casu, quo arcus AM est portio Parabolæ Conicæ, cujus natura æquatione $y^2=px$, denotante p parametrum axis, exprimitur, habetur, ex tenore formulæ generalis, in §. 3. allatæ, $A=0$, $B=p$ & $C=0$, adeoque $z=x-c+\frac{c(\frac{1}{2}B+Cx)}{((\frac{1}{2}B+Cx)^2+A+Bx+Cx^2)^{\frac{1}{2}}}$

$$=x-c+\frac{\frac{1}{2}cp}{\sqrt{\frac{1}{4}p^2+px}}=x-c+\frac{cp}{\sqrt{p^2+4px}} \;\&\; v=$$

$$\sqrt{A+Bx+Cx^2}-\frac{c\sqrt{A+Bx+Cx^2}}{((\frac{1}{2}B+Cx)^2+A+Bæ+Cx^2)^{\frac{1}{2}}}$$

$=$

$$= p^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{c p^{\frac{1}{2}} x^{\frac{1}{2}}}{\sqrt{\frac{1}{4} p^2 + px}} = p^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{2 c p^{\frac{1}{2}} x^{\frac{1}{2}}}{\sqrt{p^2 + 4px}}.$$ Has vero æquationes comparando habebitur $\sqrt{p^2 + 4px} =$

$$\frac{cp}{z - x + c} = \frac{2 cp^{\frac{1}{2}} x^{\frac{1}{2}}}{p^{\frac{1}{2}} x^{\frac{1}{2}} - v};$$ ex quibus itaque patet fore

$$\frac{cp}{z - x + c} = \frac{2 cp^{\frac{1}{2}} x^{\frac{1}{2}}}{p^{\frac{1}{2}} x^{\frac{1}{2}} - v},$$ quæ æquatio, facta debita terminorum reductione, in hanc abit formam: $x^{\frac{3}{2}} + \left(\frac{p}{2} - z - c\right) x^{\frac{1}{2}} = \frac{p^{\frac{1}{2}} v}{2}$, unde, secundum regulas pro solutionibus æquationum Cubicarum consvetas, eruitur

$$x^{\frac{1}{2}} = \left(-\frac{p^{\frac{1}{2}} v}{4} \pm \sqrt{\tfrac{1}{16} pv^2 + \tfrac{1}{27} \left(\frac{p}{2} - z - c\right)^3} \right)^{\frac{1}{3}} -$$

$$\left(\frac{p^{\frac{1}{2}} v}{4} \pm \sqrt{\tfrac{1}{16} pv^2 + \tfrac{1}{27} \left(\frac{p}{2} - z - c\right)^3} \right)^{\frac{1}{3}}$$ atque

$$x = \left(\left(-\frac{p^{\frac{1}{2}} v}{4} \pm \sqrt{\tfrac{1}{16} pv^2 + \tfrac{1}{27} \left(\frac{p}{2} - z - c\right)^3} \right)^{\frac{1}{3}} - \right.$$

$$\left. \left(\frac{p^{\frac{1}{2}} v}{4} \pm \sqrt{\tfrac{1}{16} pv^2 + \tfrac{1}{27} \left(\frac{p}{2} - z - c\right)^3} \right)^{\frac{1}{3}} \right)^2$$ ex quibus

denuo habebitur $z = \left(\left(\frac{-p^{\frac{1}{2}} v}{4} \pm \sqrt{\tfrac{1}{16} pv^2 + \tfrac{1}{27} \left(\frac{p}{2} - z - c\right)^3} \right)^{\frac{1}{3}} \right.$

$$-\left(\frac{p^{\frac{1}{2}}v}{4} \pm \sqrt{\tfrac{1}{16}pv^2 + \tfrac{1}{27}\left(\frac{p}{2} - z - c\right)^3}\right)^{\frac{1}{3}}\Bigg)^2 - c +$$

$$\frac{cp}{\left(p^2 + 4p\left(\left(-\frac{p^{\frac{1}{2}}v}{4} \pm \sqrt{\tfrac{1}{16}pv^2 + \tfrac{1}{27}\left(\frac{p}{2} - z - c\right)^3}\right)^{\frac{1}{3}} - \left(\frac{p^{\frac{1}{2}}v}{4} \pm \sqrt{\tfrac{1}{16}pv^2 + \tfrac{11}{27}\left(\frac{p}{2} - z - c\right)^3}\right)^{\frac{1}{3}}\right)^2\right)^{\frac{1}{2}}}$$

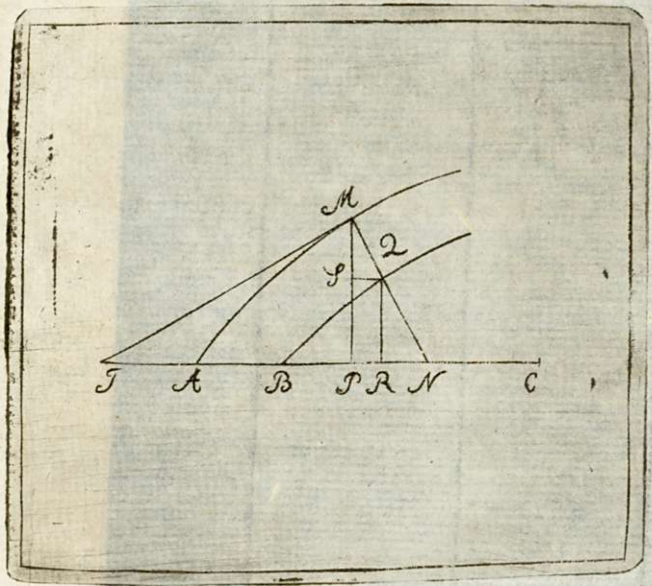
æquatio exhibens relationem inter curvæ parallelæ Coordinatas Orthogonales z & v. Hinc vero perſpicitur, Curvas Sectionibus Conicis paralleſas, excepto circulo, omnes diverſi quidem generis eſſe a curva data; quod ex ſequentibus quoque patebit.

Æquatio Curvæ BQ, quamvis maxime ſit implicita, conſtructionem tamen curvæ facillimam ſubnormalis ipſius exhibet. Capiatur nempe in curva data AM punctum quodvis M, e quo demittatur linea PM perpendiculariter in AC, & ſumatur $PN = \frac{1}{2}p$, eritque, junctis M & N, MN Normalis Parabolæ, in qua ſumta $MQ = c$; habebitur punctum Q in curva quæſita parallela. Eodemque modo alia quoque puncta determinari poſſunt, adeoque deſcriptio ipſius curvæ haud difficilis eſt cenſenda

Quod ſi vero deſideretur, curvam alteri AM datæ parallelam ita ducere, ut per punctum datum Q transeat, Normalis curvæ AM per punctum iſtud tranſiens primo determinetur, neceſſe eſt. Hoc vero problema ſequenti modo ſolui poteſt: puta factum. Sit MN Normalis curvæ AM per punctum Q tranſiens, ducatur MT ita ut curvam in M tangat, producatur PR usque ad T; & demittantur lineæ MP, QR perpendiculariter in AC, & SQ paralrallela ipsi AC. Ponatur $AR = a$, $RQ = b$, $AP = x$, $PM = y$ & parameter Parabolæ $= p$. Ob $\triangle$' $TPM \backsim \triangle\ MSQ$ erit $TP : PM :: SM : SQ$, h. e. $2x : y :: y - b : SQ = y \frac{(y-b)}{2x}$; est autem $x = AR - PR = a - y \frac{(y-b}{2x}$, unde $2x^2 = 2ax - y^2 + by$, adeoque $y = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} + 2x(a - x)}$. Ex æquatione autem Parabolæ $y^2 = px$, eruitur $y = \sqrt{px}$; comparando itaque valores ipsius y, habebitur $\sqrt{px} = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} + 2x(a - x}$ & terminis evolutis $2x(a - x) - px = - b\sqrt{px}$; cujus æquationis quadratum sumendo prodit, membris rite dispositis, $x^3 - 2(a - \frac{1}{2}p)x^2 + (a - \frac{1}{2}p)^2 x = \frac{b^2 p}{4}$, hinc vero, positis brevitatis ergo $\frac{11}{9}(a - \frac{1}{2}p)^2 = - A$ & $\frac{2}{9}(a - \frac{1}{2}p)^3 - \frac{b^2 p}{4} = B$, habebitur $x =$

$$\sqrt[3]{-\tfrac{1}{2}B \pm \sqrt{\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3}} - \sqrt[3]{\tfrac{1}{2}B \pm \sqrt{\tfrac{1}{4}B^2 + \tfrac{1}{27}A^3}} +$$

$\frac{2}{3}(a - \frac{1}{2}p)$. Determinata sic AP, facillime innotescit $PN = \frac{1}{2}p$, adeoque etiam linea MN, & sumta in axe abscis-



M
Q
S
T
A
B
P
R
N
C

ſciſſarum $AB = MQ$, erit B origo abſciſſarum curvæ parallelæ BQ, tranſeuntis per punctum Q poſitione datum. Data vero origine curvæ parallelæ & diſtantia inter vertices curvarum, facillima erit conſtructio curvæ BQ ſecundum methodum ſupra allatam.